

Smooth attractors of finite dimension for von Karman evolutions with nonlinear frictional damping localized in a boundary layer.

Pelin G. Geredeli
Hacettepe University
Ankara, Turkey
pguven@hacettepe.edu.tr

Irena Lasiecka
University of Virginia
Charlottesville, VA
il2v@virginia.edu

Justin T. Webster
University of Virginia
Charlottesville, VA
jtw3k@virginia.edu

January 31, 2012

Abstract

In this paper dynamic von Karman equations with localized interior damping supported in a boundary collar are considered. Hadamard well-posedness for von Karman plates with various types of nonlinear damping are well-known, and the long-time behavior of nonlinear plates has been a topic of recent interest. Since the von Karman plate system is of "hyperbolic type" with *critical nonlinearity* (noncompact with respect to the phase space), this latter topic is particularly challenging in the case of *geometrically constrained* and *nonlinear* damping. In this paper we first show the existence of a compact global attractor for finite-energy solutions, and we then prove that the attractor is both *smooth* and finite dimensional. Thus, the hyperbolic-like flow is stabilized asymptotically to a smooth and finite dimensional set.

Key terms: dynamical systems, long-time behavior, global attractors, nonlinear plates, nonlinear damping, localized damping

1 Introduction

We consider the evolution of a nonlinear von Karman plate subject to nonlinear frictional damping with essential support in a boundary collar. Our aim is to consider the long-time behavior of the corresponding evolution. This includes studying (a) existence of global attractor which captures long-time behavior of the dynamics, and (b) properties of this attractor, such as smoothness and finite dimensionality.

In short, our goal is to show that the original infinite dimensional and non-smooth dynamics of hyperbolic type can be reduced (asymptotically) to a finite dimensional and regular set, with respect to the topology of "finite energy". The latter is associated with weak (or generalized) solutions of the underlying semigroup for the dynamics. This type of result then allows the implementation of tools from finite dimensional control theory in order to achieve a preassigned outcome for the dynamics.

1.1 Model and Energies

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega = \Gamma$ taken to be sufficiently smooth. We consider a plate model where the real-valued function $u(x, y; t)$ models the out-of-plane displacement of a plate with negligible thickness. Then the von Karman model [16, 34] requires that u satisfies

$$\begin{aligned} u_{tt} + \Delta^2 u + d(\mathbf{x})g(u_t) &= f_V(u) + p \quad \text{in } \Omega \times (0, \infty) \equiv Q \\ u|_{t=0} &= u_0, \quad u_t|_{t=0} = u_1. \end{aligned} \quad (1.1)$$

The von Karman nonlinearity

$$f_V = [v(u) + F_0, u] \quad (1.2)$$

is given in terms of (a) the Airy Stress function $v(u)$, satisfying

$$\begin{aligned} \Delta^2 v(u) &= -[u, u] \quad \text{in } \Omega \\ \partial_\nu v(u) &= v(u) = 0 \quad \text{on } \Gamma, \end{aligned} \quad (1.3)$$

and (b) the von Karman bracket given by

$$[u, w] = u_{xx}w_{yy} + u_{yy}w_{xx} - 2u_{xy}w_{xy}. \quad (1.4)$$

The internal force $F_0 \in H^\theta(\Omega) \cap H_0^1(\Omega)$, $\theta > 3$, and external force $p \in L_2(\Omega)$ play an essential role in shaping the nontrivial stationary solutions. (In this paper $H^s(D)$ denotes the Sobolev space of order $s \in \mathbb{R}$ on domain D .) In the absence of these forces, the stationary solution of the corresponding nonlinear boundary value problem becomes trivial and simply reduces to zero.

In this treatment we focus on the stabilizing properties of the damping term $d(\mathbf{x})g(u_t)$. In particular, we take $g(\cdot) \in C(\mathbb{R})$ to be a monotone increasing function, with $g(0) = 0$ and further boundedness and smoothness assumptions to be imposed later; additionally, $d(\mathbf{x}) \equiv d_\omega(\mathbf{x})$ is a nonnegative $L_\infty(\Omega)$ localizing function which restricts the damping term $g(u_t)$ to a particular subset $\omega \subset \Omega$. This is to say $\omega \subset \text{supp } d$ or $d(x) \geq c_0 > 0$ for $x \in \omega$. Initially we will take ω to be a general set $\omega \subset \subset \Omega$, but more specifically, we are interested in taking ω to be an open collar of the boundary Γ . This type of damping represents localized, viscous damping *near* the boundary Γ .

The boundary conditions we consider for the plate are:

1. Clamped, denoted **(C)**

$$u = \partial_\nu u = 0 \quad \text{in } \Gamma \times (0, \infty) \equiv \Sigma. \quad (1.5)$$

2. Hinged (simply-supported), which we denote by **(H)**

$$u = \Delta u = 0 \quad \text{in } \Sigma. \quad (1.6)$$

3. Free-type, denoted by **(F)**

$$\begin{aligned}\mathcal{B}_1 u &\equiv \Delta u + (1 - \mu)B_1 u = 0 \quad \text{on } \Gamma_1, \\ \mathcal{B}_2 u &\equiv \partial_\nu \Delta u + (1 - \mu)B_2 u - \mu_1 u - \beta u^3 = 0 \quad \text{on } \Gamma_1, \\ u &= \partial_\nu u = 0 \quad (\text{clamped}) \quad \text{on } \Gamma_0,\end{aligned}\tag{1.7}$$

where we have partitioned the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ (with Γ_0 possibly empty). For simplicity we assume that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$. Otherwise the regularity theory for elliptic problems with *mixed* boundary conditions must be invoked. The boundary operators B_1 and B_2 are given by [34]:

$$\begin{aligned}B_1 u &= 2\nu_1 \nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx}, \\ B_2 u &= \partial_\tau \left[(\nu_1^2 - \nu_2^2) u_{xy} + \nu_1 \nu_2 (u_{yy} - u_{xx}) \right],\end{aligned}$$

where $\nu = (\nu_1, \nu_2)$ is the outer normal to Γ , $\tau = (-\nu_2, \nu_1)$ is the unit tangent vector along Γ . The parameters μ_1 and β are nonnegative, the constant $0 < \mu < 1$ has the meaning of the Poisson modulus.

Notation: Note, when referencing the plate equation above in (1.1) we will write (1.1)(C), (1.1)(H), or (1.1)(F) to indicate which boundary conditions we are taking. We write the norm in $H^s(D)$ as $\|\cdot\|_s$ and $\|\cdot\|_0 \equiv \|\cdot\|_{L_2(D)}$; for simplicity (when the meaning is clear from context) norms and inner products written without subscript $(\cdot, \cdot, \|\cdot\|)$, are taken to be $L_2(D)$ of the appropriate domain D . Additionally, we employ the notation that $H_0^s(D)$ gives the closure of $C_0^\infty(D)$ in the $\|\cdot\|_s$ norm.

The von Karman plate equation is well-known in nonlinear elasticity, and constitutes a basic model to describe the nonlinear oscillations of a thin plate with large displacements [34] (and references therein). In particular, we take the thickness of the plate to be negligible (as is usual in the modeling of thin structures [16]).

Remark 1.1. It is worth noting that the von Karman plate model can accomodate plates with non-negligible thickness - the equation in (1.1) then gives the vertical displacement of the central plane of the plate. This is tantamount to adding the term $-\gamma \Delta u_{tt}$, $\gamma > 0$ to the LHS of (1.1). This term corresponds to rotational inertia in the filaments of the plate, and (a) is *regularizing* from the energetic point of view and (b) forces the dynamics of the plate to be hyperbolic. In this treatment we take $\gamma = 0$, since it constitutes the most challenging problem mathematically, however, a future manuscript will address the case $\gamma > 0$ and the limiting problem (convergence of solutions and attractors) as $\gamma \searrow 0$.

The energies associated to the above equation are given by (in the case of clamped (C) or hinged (H) boundary conditions)

$$\begin{aligned}E(t) &= \frac{1}{2} (\|\Delta u\|^2 + \|u_t\|^2), \\ \widehat{E}(t) &= E(t) + \frac{1}{4} \|\Delta v(u)\|^2 \\ \mathcal{E}(t) &= E(t) + \Pi(u),\end{aligned}$$

where

$$\Pi(u) = \frac{1}{4} \int_{\Omega} (|\Delta v(u)|^2 - 2[F_0, u]u - 4pu). \quad (1.8)$$

The above (linear) energy $E(t)$ dictates our state space \mathcal{H} , which depends on boundary conditions. In the case of clamped boundary conditions (C) we have $\mathcal{H}_1 \equiv H_0^2(\Omega) \times L_2(\Omega)$. For hinged boundary conditions (H) we have $\mathcal{H}_2 \equiv (H^2 \cap H_0^1)(\Omega) \times L_2(\Omega)$.

Lastly, for free boundary conditions (F) we have $\mathcal{H}_3 \equiv (H^2 \cap H_{0,\Gamma_0}^2)(\Omega) \times L_2(\Omega)$ (where $H_{0,\Gamma_0}^2(\Omega)$ is the Sobolev space $H^2(\Omega)$ with clamped conditions on Γ_0); the potential energy in this case is given by the bilinear form

$$a(u, v) = \int_{\Omega} \tilde{a}(u, v) + \mu_1 \int_{\Gamma_1} uv, \quad (1.9)$$

where

$$\tilde{a}(u, v) \equiv u_{xx}v_{xx} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1 - \mu)u_{xy}v_{xy}. \quad (1.10)$$

Then the energy becomes

$$E(t) = \frac{1}{2} \{ \|u_t\|^2 + a(u(t), u(t)) \}$$

$$\widehat{E}(t) \equiv E(t) + \frac{1}{4} \|\Delta v(u)\|^2 + \frac{\beta}{2} \int_{\Gamma_1} u^4 d\Gamma.$$

The total energy becomes

$$\mathcal{E}(t) = E(t) + \Pi(u(t)) + \frac{1}{4}\beta \int_{\Gamma_1} u^4(t).$$

Remark 1.2. We note that this last form of the energy described by the bilinear form $a(u, v)$ can also be applied to clamped or hinged boundary conditions. Indeed, in this latter case the bilinear form $a(u, u)$ collapses just to $\|\Delta u\|^2$.

It will be convenient to introduce an elliptic operator $A: \mathcal{D}(A) \subset L_2(\Omega) \rightarrow L_2(\Omega)$ given by $Au = \Delta^2 u$, where $\mathcal{D}(A)$ incorporates the corresponding boundary conditions (clamped, hinged, or free). It is useful to note that by elliptic regularity

$$\mathcal{D}(A^{1/2}) = \begin{cases} H_0^2(\Omega) & \text{clamped BC} \\ (H^2 \cap H_0^1)(\Omega) & \text{hinged BC} \\ (H^2 \cap H_{0,\Gamma_0}^2)(\Omega) & \text{free BC} \end{cases}$$

It is important to note the total potential energy may not be positive, or even not bounded from below. This is due to the presence of internal force F_0 which may drive the energy to $-\infty$. However, the presence of the von Karman bracket in the model, along with appropriate regularity properties imposed on F_0 , assures that the energy is bounded from below. This can be seen from the following lemma [12, 13]:

Lemma 1.1. *Let $u \in \mathcal{D}(A^{1/2})$, $p \in L_2(\Omega)$, and $F_0 \in H_0^1(\Omega) \cap H^\theta(\Omega)$, $\theta > 3$. Then, $\forall \epsilon > 0$ there exists $M(\epsilon, \|p\|, \|F_0\|_\theta) = M_{\epsilon,p,F_0} < \infty$ such that in the clamped and hinged case*

$$\|u\|^2 \leq \epsilon (\|A^{1/2}u\|^2 + \|\Delta v(u)\|^2) + M_{\epsilon,p,F_0}$$

and in the free case with $\beta > 0$,

$$\|u\|^2 \leq \epsilon(\|\mathcal{A}^{1/2}u\|^2 + \|\Delta v(u)\|^2 + \frac{\beta}{2}\|u\|_{L^4(\Gamma)}^4) + M_{\epsilon,p,F_0,\beta}$$

As a consequence we have the following bounds from below for the energy:
There exist positive constants m, c, M, C such that

$$-m + c\widehat{E}(t) \leq \mathcal{E}(t) \leq M + C\widehat{E}(t) \quad (1.11)$$

$$-m + cE(t) \leq \mathcal{E}(t) \leq h(E(t)) \quad (1.12)$$

where $h(s)$ denotes a continuous function.

1.2 Motivation and Literature

Well-posedness for von Karman's plate equation with interior and/or boundary dissipation has been known for some time for smooth solutions in the case of homogeneous [8] or inhomogeneous nonlinear boundary conditions [12, 21] and references therein. The issue of well-posedness for 'weak' (finite-energy) solutions is more recent [12, 21]. In this paper, we are interested in homogeneous type boundary conditions and we will be considering *generalized* nonlinear semigroup solutions [4, 43] which also can be shown to be *weak* variational solutions. For a detailed and complete discussion regarding the wellposedness and regularity of von Karman solutions the reader is referred to [12, 28]. In the context of this paper we will need the following well-posedness result, which is contingent upon the recently shown sharp regularity of the Airy Stress function in (1.3) [21, 12]:

Theorem 1.2. *With reference to problem 1.1(C) with initial data $(u_0, u_1) \in \mathcal{H}_1$, or 1.1(H) with initial data $(u_0, u_1) \in \mathcal{H}_2$, or 1.1(F) with initial data $(u_0, u_1) \in \mathcal{H}_3$, there exists a unique global solution of finite-energy (i.e. $(u, u_t) \in C([0, T]; \mathcal{H}_i)$ for $i = 1, 2, 3$ resp., for any $T > 0$). Additionally, (u, u_t) depends continuously on $(u_0, u_1) \in \mathcal{H}_i$.*

Thus, for any initial data in the finite energy space $(u_0, u_1) \in \mathcal{H}$, there exists a well defined semiflow (nonlinear semigroup) $S_t(u_0, u_1) \equiv (u(t), u_t(t)) \in \mathcal{H}$ which varies continuously with respect to the initial data in \mathcal{H} . The domain of the corresponding generator $\mathcal{A}(u, v) \equiv (v, -Au - d(\mathbf{x})g(v) + f_V(u) + p)$ is given by $\mathcal{D}(\mathcal{A}) = \{(u, v) \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}); Au + d(\mathbf{x})g(v) \in L_2(\Omega)\}$. For initial data taken in $\mathcal{D}(\mathcal{A})$, the corresponding solutions are regular and remain invariant in $\mathcal{D}(\mathcal{A})$ [4, 39, 43]. With an additional assumption that $g(s)$ is bounded polynomially at infinity, one has $\mathcal{D}(\mathcal{A}) \subset H^4(\Omega) \times H^2(\Omega)$. Equipped with the regularity of the domain $\mathcal{D}(\mathcal{A})$, one derives the energy identity for all regular solutions. Due to the density of the embedding $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$, monotonicity of the damping, and sharp regularity of the Airy stress function (see Lemma 3.4) the same energy equality remains valid for all generalized solutions corresponding to any boundary conditions under consideration. Thus we have the energy identity for boundary conditions (C), (H), or (F) satisfied for all generalized (semigroup) solutions (complete details of this argument are given in [12]).

This equality reads: for all $0 < s < t$, strong and generalized solutions u to (1.1) satisfy

$$\mathcal{E}(t) + \int_s^t \int_{\Omega} d(\mathbf{x})g(u_t)u_t = \mathcal{E}(s) \quad (1.13)$$

With the well-posedness of the semiflow established in Theorem 1.1, it is natural to investigate long time behavior of the dynamical system generated by (1.1). It is clear from (1.13) that the essential mechanism for dissipating the energy is the damping term $d(x)g(u_t)$. In the simplest possible scenario when $p = F_0 = 0$ the energy function $\mathcal{E}(t)$ is equivalent topologically to the norm of the phase space \mathcal{H} . Since $\mathcal{E}(t)$ is nonincreasing on the trajectories, it becomes a Lyapunov function for the corresponding nonlinear dynamical system, whose only equilibrium is the zero point. If one assumes that $d(x) > 0$, *a.e.* in Ω , it is well known that $\mathcal{E}(t)$ becomes a *strict* Lyapunov function and zero equilibrium is strongly stable. However, the above condition imposed on $d(x)$ is *not sufficient* to guarantee uniform convergence to the equilibrium (this is also the case for *linear* dynamics without the von Karman term). In order to secure uniform convergence or, more generally, convergence to a compact attractor, a stronger form of the damping is necessary. For example, $d(x) \geq c_0 > 0$, $x \in \Omega$ and $g(s) = as$, $a > 0$, provides a classical model for which uniform convergence to zero in the absence of external/internal forcing (or more generally to an attractor) can be shown [7, 8, 12, 34]. The goal in this paper is to consider nonlinear damping of a reduced essential support whereby the inequality $d(x) \geq c_0 > 0$ will be enforced only in a small set $\omega \subset\subset \Omega$, while the dynamics will be forced by nontrivial sources p , F_0 . Existence of a compact and possibly smooth finite dimensional attracting set for the dynamics generated by (1.1) with boundary conditions (C), (H), or (F) and geometrically constrained dissipation is of great physical interest. Such a result is tantamount to asserting that the infinite dimensional, non-smooth dynamics are asymptotically reduced to a *smooth and finite dimensional set*. While such a reduction is expected for dynamical systems that exhibit some smoothing effects (e.g. parabolic-like) [44, 37, 17, 41, 2, 38, 26], it is a much less evident phenomena in the case of hyperbolic-like dynamics, where the ‘taking-off’ of the dynamics produces no smoothing effect. The role of the frictional damping in such a system is instrumental; in fact, it is the induced friction that creates a stabilizing and asymptotically regularizing effect on the evolution, ultimately reducing it to a compact set. On the other hand it is well known that the hyperbolic-like dynamics can not be stabilized by a compact feedback operator [30] (and references therein). This is due to the fact that instabilities in the system are inherently infinite dimensional and the essential part of the spectrum can not be dislodged by a compact perturbation. Thus, any effective damping cannot be compact (with respect to the phase space). The above feature combined with (a) nonlinearity of the damping and (b) lack of compactness of the nonlinear von Karman source makes the analysis of long-time behavior for this class of systems challenging. In fact, critical exponent nonlinearities and nonlinear dissipation are known to constitute endemic difficulties in the study of hyperbolic-like systems [22].

To orient the reader and to provide some perspective for the problem studied, we shall briefly describe some of the principal contributions to this area of research. A detailed account is given in [12].

In the discussion of global attractors for von Karman evolution equations, we must distinguish between two types of dynamics for the problem: (a) the rotational case (as addressed above) when the term $-\gamma\Delta u_{tt}$, $\gamma > 0$ is added to the LHS of (1.1) and (b) nonrotational ($\gamma = 0$). In case (a), we note that the von Karman nonlinearity (in the finite energy topology) is *compact*, which considerably simplifies the analysis of long-time dynamics. In the latter case (b) (which we consider here), a very different type of analysis is needed. Here, we shall focus on part (b) only. In fact, the very first contribution to this problem is a pioneering paper [8] where the existence of *weak* attractors with a

linear, fully supported damping was demonstrated. Later on, owing to new results on the regularity of Airy's stress function [21, 12], *weak* attractors were improved to *strong* attractors, and the restriction of linear damping was removed in order to allow nonlinear, monotone damping [11]. In order to incorporate fully nonlinear interior damping, [11] assumes that the dissipation parameter is sufficiently *large*. This restriction was later removed in [27], whose paper introduces a very clever way of bypassing a lack of compactness and replacing it with an "iterated convergence" trick. Further studies of the attractor (including properties such as dimensionality and smoothness) in the fully nonlinear setup, without "size" restrictions imposed on the parameters, are presented in [14] and in monograph form in [10, 12].

It should be noted that the results described above pertain to the interior and *fully supported* dissipation. The situation is much more delicate when the dissipation is *geometrically constrained*, where the essential support of the damping is localized to a *subset* of the spatial domain Ω . In that case, the issue of propagating the damping from one area to another becomes the critical one. While this sort of problems has been previously studied in the context of stabilization to equilibria [34, 24, 25], the estimates needed for attractors are much more demanding. Previous methods developed in the context of stabilization no longer apply. Some long-time behavior results with boundary damping are presented in [13, 14], wherein nonlinear dissipation on the boundary acting via *free* boundary conditions is considered. These works, however, impose the rather stringent geometric restrictions of the entire boundary being star-shaped. Such restrictions are removed in [12], where dissipation via hinged boundary conditions is considered; however this is done at the expense of limiting the class of dissipation to those of *linearly bounded* type. This restriction is needed since the elimination of the geometric condition is achieved via microlocal estimates [32], which in turn force velocity dependent nonlinear terms to be linearly bounded.

This brings us to the main contribution of the present manuscript. Our goal is to show that the fully *nonlinear* damping with essential support in an arbitrarily small layer near the boundary provides not only the existence of compact attractors but also desirable properties such as smoothness and finite dimensionality. Thus the original hyperbolic-like non-smooth flow is asymptotically reduced to smooth and finite dimensional dynamics. The result is valid for *all types of boundary conditions* with *geometrically constrained dissipation*, which can be nonlinear of *any polynomial growth* at infinity and with *no restriction on the size* of the damping parameter.

We obtain this result by proving that the dynamics are *quasi-stable* - a concept introduced in [10] and [12]. The ability to show quasi-stability is dependent upon: (a) a new method of localization of multipliers that allows smooth propagation of the damping from the boundary collar into the interior (even in the presence of boundary conditions (free) that do not comply with the Lopatinski conditions [42]) and (b) "backward" smoothness of trajectories from the attractor - a method used also in [14] and in [15, 5] - the latter in the context of geometrically constrained dissipation for wave dynamics.

Lastly, we would like to note that while some of the methods developed for *boundary* dissipation [12, 14] can also be used in the case of partially localized dissipation and Dirichlet - clamped boundary conditions, this is not the case with Neumann type (free) boundary conditions which violate strong Lopatinski [42] condition. In this latter case, propagation of the damping from the boundary layer via boundary damping estimates is obstructed by the well known lack of sufficient regularity (the absence of so called "hidden" regularity [31]) of boundary traces corresponding to the linear model [32]. Our aim in this paper is to develop a method which is effective for all kind of boundary conditions and does

not depend on hidden regularity, where the latter restricts the analysis to Lopatinski type of models. The key element for this are suitably localized multipliers estimates.

1.3 Statement of Results

Equipped with well-posedness of finite energy and regular solutions corresponding to (1.1) under one of the boundary conditions (C), (H), or (F), we are now ready to state our main results pertaining to long time behavior of solutions. In order to do this, we shall introduce the following unique continuation condition, denoted *UC*. We say that the system satisfies the *UC* property iff the following implication is valid for any weak solution (u, u_t) to (1.1): There exists $T > 0$ such that

$$u_t = 0 \text{ a.e. in } \text{supp } d \times (0, T) \Rightarrow u_t = 0 \text{ a.e. in } \Omega \times (0, T).$$

It is clear that the *UC* property holds if $d(\mathbf{x}) > 0$ a.e. in Ω .

Remark 1.3. It is worth noting that due to the non-local character of von Karman nonlinearity, the unique continuation property for the von Karman plate is not fully understood. A now classical set of tools developed for plate equations and based on Carleman estimates [18, 1, 20, 19] do not apply. The non-locality of the von Karman bracket prevents propagation across the entire domain of *weak* damping localized to a small set. Therefore, we have the question: *if the damping in the equation (represented by $d(\mathbf{x})u_t$) is zero in an open set of positive measure inside of Ω , does this imply that the solution u must also be 0 in Ω ?* it remains open. In relation to our analysis here, if the general unique continuation property holds for the von Karman plate, then it immediately strengthens our result by allowing $d(\mathbf{x})$ to vanish away from an open collar of the boundary. However, at present, the best we can state is a sufficient condition, namely that $d(\mathbf{x}) > 0$ a.e. in order to satisfy the *UC* property.

In addition, we shall assume validity of an *asymptotic* growth condition *from below* imposed on $g(s)$. Such condition is typical [34] and necessary in order to obtain uniform decay rates of solutions in hyperbolic-like dynamics. It allows control of the kinetic energy for large frequencies.

Assumption 1. *There exist positive constants $0 < m \leq M < \infty$ and a constant $p \geq 1$ such that*

$$m \leq g'(s) \leq M|s|^p, \quad |s| \geq 1$$

We now state the primary result in this treatment.

Theorem 1.3. *Take Assumption 1 to be in force. Let $\text{supp } d \supset \omega$ and $d(\mathbf{x}) \geq \alpha_0 > 0$ in ω , where $\omega \subset \subset \Omega$ is any full collar near the boundary Γ . Then for all generalized solutions corresponding to solutions with initial data $\|(u_0, u_1)\|_{\mathcal{H}} \leq R$, there exist compact attractor $\mathbf{A}_R \in \mathcal{H}$. If, in addition, the *UC* property holds, then said attractor is global, i.e $\mathbf{A}_R = \mathbf{A}$ for all $R > 0$.*

Remark 1.4. The *UC* property is needed in order to construct a *strict* Lyapunov function for the plate. In the absence of this Lyapunov function, the obtained results are of local character - as in the first part of Theorem 1.3. The *UC* property allows us to conclude that local attractors coincide with a global one.

Theorem 1.4. *In addition to Assumption 1 and the *UC* property, assume that there exists $m, M > 0$, and $\gamma < 1$ such that $0 < m \leq g'(s) \leq M[1 + sg(s)]^\gamma$, for all $s \in \mathbb{R}$. Then,*

- (a) the attractor \mathbf{A} is regular, which is to say $\mathbf{A} \subset H^4(\Omega) \times H^2(\Omega)$ is a bounded set in that topology.
- (b) The fractal dimension of \mathbf{A} is finite.

Remark 1.5. If we consider $g(s) = |s|^{p+1}$, then one can show that $\gamma = \frac{p}{p+2}$ satisfies the above condition.

There are three main difficulties/novelities pertaining to the proof of the results stated above:

- (a) The nonlinear source is of critical exponent (lack of compactness).
- (b) The essential damping is geometrically constrained to a small subset ω .
- (c) The damping is genuinely non-linear (any polynomial growth at the infinity is allowed).

These three difficulties are well-recognized in the context of studying long time behavior of hyperbolic-like systems where there is no inherent smoothing mechanism present in the model. In order to provide some perspective, it helps to add that geometrically constrained damping forces to use higher order multipliers which become *supercritical* when dealing with energy terms and nonlinear critical terms. Thus, any successful approach must rely on suitable cancellations, which must be uncovered for the specific dynamics in question.

Similar issues appear when dealing with nonlinear damping. The damping term must be critical (in hyperbolic dynamics) in order to be effective (we recall that the essential spectrum of an operator can not be altered by a compact perturbation). The property of monotonicity of the problem does help when dealing with a single solution at the energy level. However, when dealing with long-time behavior, the protagonist is not a single solution but the difference of two solutions. In the study of the corresponding dynamics at the non-energetic levels (resulting from multipliers), monotonicity is destroyed. There is a “spillover” of the noncompact (in fact, supercritical) damping that must be absorbed. For this issue, different mechanisms need to be discovered (e.g. backward smoothness of trajectories, compensated compactness, etc).

While recent developments in the field provide tools enabling us to handle a combination of *any two* of the difficulties listed above, the *inclusion of the third* prevents us from utilizing existing mathematical technology. The principal contribution of this treatment is to develop method which is capable of dealing with all three aforementioned difficulties simultaneously. The main ingredients of this new approach are (i) a localization method which allows us to show propagation of the damping without any requiring that the Lopatinski condition be satisfied, and (ii) backward smoothness of trajectories from the attractor with geometrically localized dissipation.

We conclude this section by listing few problems that are of interest to pursue and still open.

(1) C^∞ smoothness of attractors in the presence of nonlinear damping

Regarding the first item, C^∞ smoothness of an attractor can be proved under certain restrictions on nonlinear damping by methods developed in [12] and also in an influential paper [23]. The treatment of a fully nonlinear and monotone dissipation is still not fully understood.

(2) Damping restricted to a portion of an open collar

Secondly, dissipation localized to part of the collar could be considered by assuming certain geometric conditions imposed on the uncontrolled part of the collar. Certain ideas presented in [15, 5] should prove useful.

(3) The UC property for a larger class of dampings

Lastly, the third item is an wide open and important problem. Other mild forms of dissipation - such as viscoelastic weak damping - are also valid alternatives, however, a full understanding of

this problem would require the development of a substitute for Carleman estimates, which are not applicable due to localized structure of the von Karman bracket. The associated problem is related to controlling low frequencies - an endemic problem principally associated with strong stability [30, 29]. It is worth noting that this problem is non-existent in the case of other nonlinear plates, such as semilinear plates with local nonlinearities or even Berger's model [6], where some forms of Carleman's estimates do apply [1, 18].

2 Long-time Behavior of Dynamical Systems

In this manuscript we will make ample use dynamical systems terminology (see [2, 38, 41, 7, 12]); let (\mathcal{H}, S_t) be a dynamical system with $\mathcal{N} \equiv \{x \in \mathcal{H} : S_t x = x \text{ for all } t \geq 0\}$ the set of its stationary points.

We say that a dynamical system is *asymptotically compact* if there exists a compact set K which is uniformly attracting: for any bounded set $D \subset \mathcal{H}$ we have that

$$\lim_{t \rightarrow +\infty} d_{\mathcal{H}}\{S_t D | K\} = 0 \quad (2.1)$$

in the sense of the Hausdorff semidistance.

(\mathcal{H}, S_t) is said to be *asymptotically smooth* if for any bounded, forward invariant ($t > 0$) set D there exists a compact set $K \subset \overline{D}$ such that (2.1) holds. An asymptotically smooth dynamical system should be thought of as one which possesses *local attractors*, i.e. for a given ball $B_R(x)$ of radius R in the space \mathcal{H} there exists a compact attracting set in the closure of $B_R(x)$, however, this set need not be uniform with respect to R or $x \in \mathcal{H}$.

A *global attractor* \mathbf{A} is a closed, bounded set in \mathcal{H} which is invariant (i.e. $S_t \mathbf{A} = \mathbf{A}$ for all $t > 0$) and uniformly attracting (as defined above).

A *strict Lyapunov function* for (\mathcal{H}, S_t) is a functional Φ on \mathcal{H} such that (a) the map $t \rightarrow \Phi(S_t x)$ is nonincreasing for all $x \in \mathcal{H}$, and (b) $\Phi(S_t x) = \Phi(x)$ for all $t > 0$ and $x \in \mathcal{H}$ implies that x is a stationary point of (\mathcal{H}, S_t) . If the dynamical system has a strict Lyapunov function, then we say that (\mathcal{H}, S_t) is *gradient*.

In the context of this paper we will use a few key theorems (which we now formally state) to prove the existence of the attractor. (For proofs and references, see [12] and references therein.) First, we address attractors for gradient systems and characterize the attracting set:

Theorem 2.1. *Suppose that (\mathcal{H}, S_t) is a gradient, asymptotically smooth dynamical system. Suppose its Lyapunov function $\Phi(x)$ is bounded from above on any bounded subset of \mathcal{H} and the set $\Phi_R \equiv \{x \in \mathcal{H} : \Phi(x) \leq R\}$ is bounded for every R . If the set of stationary points for (\mathcal{H}, S_t) is bounded, then (\mathcal{H}, S_t) possesses a compact global attractor \mathbf{A} which coincides with the unstable manifold, i.e.*

$$\mathbf{A} = \mathcal{M}^u(\mathcal{N}) \equiv \{x \in \mathcal{H} : \exists U(t) \in \mathcal{H}, \forall t \in \mathbb{R} \text{ such that } U(0) = x \text{ and } \lim_{t \rightarrow -\infty} d_{\mathcal{H}}(U(t) | \mathcal{N}) = 0\}.$$

Secondly, we state a useful criterion (inspired by [27]) which reduces asymptotic smoothness to finding a suitable functional on the state space with a compensated compactness condition:

Theorem 2.2. *Let $(\mathcal{H}, S(t))$ be a dynamical system, \mathcal{H} Banach with norm $\|\cdot\|$. Assume that for any bounded positively invariant set $B \subset \mathcal{H}$ and for all $\epsilon > 0$ there exists a $T \equiv T_{\epsilon, B}$ such that*

$$\|S_T x_1 - S_T x_2\|_{\mathcal{H}} \leq \epsilon + \Psi_{\epsilon, B, T}(x_1, x_2), \quad x_i \in B$$

with Ψ a functional defined on $B \times B$ depending on ϵ, T , and B such that

$$\liminf_m \liminf_n \Psi_{\epsilon, T, B}(x_m, x_n) = 0$$

for every sequence $\{x_n\} \subset B$. Then (\mathcal{H}, S_t) is an asymptotically smooth dynamical system.

In order to establish both smoothness of the attractor and finite dimensionality a stronger estimate on the difference of two flows is needed.

Theorem 2.3. *Let $x_1, x_2 \in B \subset \mathcal{H}$ where B is a forward invariant set for the flow $S_t x_i$. Assume that the following inequality holds for all $t > 0$ with positive constants $C_1(B), C_2(B), \omega_B$*

$$\|S_t x_1 - S_t x_2\|_{\mathcal{H}}^2 \leq C_1(B) e^{-\omega_B t} \|x_1 - x_2\|_{\mathcal{H}}^2 + C_2(B) \max_{\tau \in [0, t]} \|S_{\tau} x_1 - S_{\tau} x_2\|_{\mathcal{H}_1}^2 \quad (2.2)$$

where $\mathcal{H} \subset \mathcal{H}_1$ is compactly embedded. Then the attractor \mathbf{A} associated with the flow S_t possesses the following properties:

- (a) *The fractal dimension of \mathbf{A} is finite.*
- (b) *For any $x \in \mathbf{A}$ one has $\frac{d}{dt}(S_t x) \in C(R, \mathcal{H})$.*

Remark 2.1. The estimate in (2.2) is often referred to as a “quasistability” estimate. It reflects the fact that the flow can be stabilized exponentially to a compact set. Alternatively, we might say that the flow is exponentially stable, modulo a compact perturbation (lower order terms). We assert that the lower order terms being quadratic is important for the validity of Theorem 2.3.

The proof of Theorem 2.3 employs the idea of “piecewise” trajectories introduced in [35, 40]. This allows to generalize previous criteria for finite-dimensionality [2, 44, 41, 17] by reducing the problem to validity of quasistable estimate.

2.1 Approach and Outline of the Paper

To show our main result on the existence of the global attractor for (1.1) with boundary conditions (C), (H), or (F) we make use of the theorems above. First, we note that in the case of any boundary conditions, the von Karman system in (1.1) is gradient with Lyapunov function $\mathcal{E}(t)$ (under the assumption that the UC property is satisfied, e.g. in the case that $d(\mathbf{x}) > 0$ a.e. in Ω). We refer to [12] for the details. Moreover, the set of stationary points for the dynamical system generated by (1.1)(C), (1.1)(H), or (1.1)(F) is bounded. This latter fact follows from (1.11) (see [12]). Hence we are in a position to use Theorem 2.1 if we can obtain an inequality of the form in Theorem 2.2 to show asymptotic smoothness of the system. We will analyze z , taken to be the difference of strong solutions, and make use of the linear energy $E_z(t) = \|\Delta z\|^2 + \|z_t\|^2$ (then, via a standard limiting procedure obtain our estimate for generalized solutions as well); this estimation will produce our functional Ψ in

Theorem 2.2. Our main tool in estimating $E_z(t)$ will be the use of two multipliers: $f_1 z$ and $h \cdot \nabla((f_2)z)$, where h will be a suitably chosen C^2 vector field and f_i are appropriate localization functions.

First, we perform multiplier analysis as generally as possible, without imposing boundary conditions. Later on, we shall use boundary conditions (either clamped, or hinged or free) in order to obtain the smoothness inequality in Theorem 2.2.

After establishing the existence of the attractor, we proceed to show that it has additional regularity than that of the state space, and also that it has finite fractal dimension. The ultimate goal is to prove a “quasistability” estimate for the difference of general trajectories and apply abstract Theorem 2.3, however doing so directly in this case is difficult. Instead, we will prove a modified quasistability estimate (via similar methods in the asymptotic smoothness calculation) which applies to the difference of two terms along the same trajectory (i.e. the difference of $u(t+h) - u(t)$). This term is substantially easier to analyze, since we have continuity in t in \mathcal{H} and both $u(t+h)$ and $u(t)$ converge to the *same* point of equilibrium for $t \rightarrow \infty$ and $t \rightarrow -\infty$. Our modified quasistability estimate hinges upon the bounding terms being quadratic, so upon division by h and taking $h \searrow 0$ we can show additional regularity of elements from the attractor for sufficiently negative times $T \ll 0$. Proving this will depend upon a trajectory being ‘close’ to a point of equilibrium, and hence yielding ‘smallness’ of the velocity of the solution. Propagating this regularity forward via the dynamical systems property will then allow us to show the additional regularity of the attractor. We will then proceed in a standard fashion to show that this additional regularity of the attractor yields the true and sought after quasistability estimate in Theorem 2.3, which will produce the finite fractal dimension of the attractor.

3 Asymptotic Smoothness

In this section we prove that the dynamical system generated by (1.1) is asymptotically smooth. We will refrain from imposing boundary conditions until absolutely necessary in the hope of unifying the treatment of (C), (H), and (F).

Note that the new variable $z = u - w$, where $(u(t), u_t(t)) = S_t(u_0, u_1)$, $(w(t), w_t(t)) = S_t(w_0, w_1)$ are solutions to (1.1) with initial data taken in bounded set in $B \subset \mathcal{H}$. On the strength of Lemma 1.1 and (1.11) we may assume that there exists $R > 0$ such that

$$\|(u(t), u_t(t))\|_{\mathcal{H}} \leq R, \quad \|(w(t), w_t(t))\|_{\mathcal{H}} \leq R, \quad t > 0 \quad (3.1)$$

The difference of two trajectories $z = u - w$ solves the following PDE:

$$\begin{aligned} z_{tt} + \Delta^2 z + \mathcal{G}(z) + \mathcal{F}(z) &= 0 \quad \text{in } Q, \\ z(0) &= u_0 - w_0; \quad z_t(0) = u_1 - w_1 \end{aligned} \quad (3.2)$$

where

$$\mathcal{F}(z) \equiv -(f_V(u) - f_V(w)), \quad \text{and } \mathcal{G}(z) \equiv d(\mathbf{x})(g(u_t) - g(w_t))$$

The above evolution is equipped with appropriate boundary conditions (C), (H), or (F) which will be specified later.

3.1 Multipliers

Ultimately, we will need a pointwise bound (in time) on the functional $E_z(t)$ as defined above. To achieve this bound, we will employ multiplier methods based on specially chosen cut-off functions λ and μ . These functions are taken to be $C^\infty(\Omega)$. Later, we will choose the supports of the derivatives of λ and μ to be contained in the damping region ω , where the damping $g(u_t)$ is effectively localized; the cut-off functions will be chosen in this way so as to reconstruct the full energy $E_z(t)$ via the multipliers, bounded in terms of the damping. However, for now, we can consider $\text{supp } \lambda \subset \Omega$ to be arbitrary.

We define the variables $\phi = \lambda z$ and $\psi = \mu z$. The use of the cut-off functions will produce commutators active in the regions of ω where the cut-off functions are non-constant. Lastly, we will make use of the following notational conventions. First, to describe (a) lower order terms:

$$l.o.t.^f \equiv \sup_{[0,T]} \|f(t)\|_{2-\eta}^2, \quad l.o.t.^f_1 \equiv \sup_{[0,T]} \|f(t)\|_{2-\eta},$$

where $0 < \eta < 1/2$, and (b) boundary terms: $B.T.^f = \left\{ \Delta f \partial_\nu f - \partial_\nu(\Delta f) f \right\}$

Remark 3.1. We note that the use of different notations for lower order terms is necessary in the handling of dissipation estimates. Specifically, we must treat the dissipation terms differently when dealing with asymptotic smoothness type estimates, and the estimates which will ultimately yield the quasistability estimate.

3.1.1 ϕ Multiplier

Let P and Q be two differential operators. We will make use of the commutator symbol given by

$$[P, Q]f = P(Qf) - Q(Pf),$$

We shall work with smooth solutions guaranteed by Theorem 1.2. Multiplying the PDE in (3.2) by λ we arrive at

$$\phi_{tt} + \Delta^2 \phi + \lambda \mathcal{G}(z) + \lambda \mathcal{F}(z) = [\Delta^2, \lambda]z.$$

Now, we employ the multiplier ϕ . This is an equipartition multiplier which allows us to reconstruct the difference between the potential and kinetic energies. The following Green's identities are available [34] for sufficiently smooth functions z and ϕ :

$$\begin{cases} \int_{\Omega} \Delta^2 z \phi = \int_{\Omega} \Delta z \Delta \phi + \int_{\Gamma} (\partial_\nu \Delta z \phi - \Delta z \partial_\nu \phi), & \text{clamped and hinged B.C} \\ \int_{\Omega} \Delta^2 z \phi = a(z, \phi) + \beta \int_{\Gamma_1} z^3 \phi + \int_{\Gamma_1} (\mathcal{B}_2 z \phi - \mathcal{B}_1 z \partial_\nu \phi), & \text{free B.C} \end{cases}$$

Using the first formula for clamped or hinged boundary conditions yields:

$$\int_Q \{ |\Delta \phi|^2 - |\phi_t|^2 \} = \int_Q [\Delta^2, \lambda] z \phi - \int_Q \lambda \{ \mathcal{G}(z) + \mathcal{F}(z) \} \phi + \int_{\Sigma} \{ \Delta \phi \partial_\nu \phi - \partial_\nu(\Delta \phi) \phi \} - (\phi_t, \phi) \Big|_0^T \quad (3.3)$$

Making use of standard splitting and Sobolev embeddings, we arrive at

$$\int_0^T \{ \|\Delta\phi\|^2 - \|\phi_t\|^2 \} \leq \int_{\Sigma} B.T.\phi + \int_Q ([\Delta^2, \lambda]z)\phi + \int_Q \lambda \{ \mathcal{G}(z) + \mathcal{F}(z) \} \phi + C(E(T) + E(0)) \quad (3.4)$$

In the case of free boundary conditions, the equipartition of energy takes the form

$$\begin{aligned} \int_0^T \{ a(\phi, \phi) + \beta |\phi|_{L^4(\Gamma)}^4 - \|\phi_t\|^2 \} &\leq \int_{\Sigma_1} (\mathcal{B}_1\phi\phi - \mathcal{B}_2\phi\partial_\nu\phi) + \int_Q ([\Delta^2, \lambda]z)\phi \\ &+ \int_Q \lambda \{ \mathcal{G}(z) + \mathcal{F}(z) \} \phi + C(E(T) + E(0)) \end{aligned} \quad (3.5)$$

We note for all boundary conditions (C), (H), the boundary terms $B.T.\phi \equiv 0$. In the free case (F) we have $\mathcal{B}_1\phi = 0$, $\mathcal{B}_2\phi = 2\beta\phi u\nu$ where the latter term contributes a lower order term to the estimate.

To continue with our observability estimation, we must explicitly bound the commutator $\int_Q [\Delta^2, \lambda]z\phi$. Purely algebraic calculations give

$$\begin{aligned} [\Delta^2, \lambda]f &= \Delta^2(\lambda f) - \lambda \Delta^2 f \\ &= (\Delta^2 \lambda)f + 2\Delta\lambda\Delta f + 2(\nabla\lambda, \nabla(\Delta f)) + 2(\nabla(\Delta\lambda), \nabla f) + 2\Delta(\nabla\lambda\nabla f) \end{aligned} \quad (3.6)$$

The calculation above implies that the commutator $[\Delta^2, \lambda]$ is a differential operator of order three. In order to exploit this in the calculations with the energy, we need to reduce the order of differential operator acting on a solution via integration by parts. This is done below.

This computation makes sole use of Green's theorem. For the sake of exposition, we do not impose any boundary conditions:

$$\int_{\Omega} \nabla \Delta u (\phi \nabla \lambda) = - \int_{\Omega} (\Delta u) \operatorname{div}(\phi \nabla \lambda) + \int_{\Gamma} (\phi \Delta u) \nabla \lambda \cdot \nu \quad (3.7)$$

$$\int_{\Omega} \Delta(\nabla \lambda \nabla u) \phi = - \int_{\Omega} \nabla(\nabla \lambda \nabla u) \nabla \phi + \int_{\Gamma} \partial_\nu(\nabla u \nabla \lambda) \phi \quad (3.8)$$

Note that due to the fact that the support of $\nabla \lambda$ is away from the boundary, all of the boundary terms in the above expressions (3.7) and (3.8) will vanish. Moreover,

$$\left| \int_{\Omega} \nabla \lambda \nabla \Delta u \phi \right| + \left| \int_{\Omega} \Delta(\nabla \lambda \nabla u) \phi \right| \leq C_\lambda \|u\|_2 \|\phi\|_1 \quad (3.9)$$

Hence to conclude our ϕ multiplier estimate, we have the following technical lemma:

Lemma 3.1 (Preliminary ϕ Estimate). *Let $\phi \equiv \lambda z$, as defined above, where z solves (3.2) with boundary conditions (C) or (H). Then, there exists $0 < C < \infty$ such that*

$$\int_0^T \{ \|\Delta\phi\|^2 - \|\phi_t\|^2 \} \leq C(T, \lambda) l.o.t.^z + \int_Q \lambda \{ \mathcal{G}(z) + \mathcal{F}(z) \} \phi + C(E_z(T) + E_z(0)) \quad (3.10)$$

In the free case (F)

$$\int_0^T \{a(\phi, \phi) + \beta \int_{\Gamma} \phi^4 - \|\phi_t\|^2\} \leq C(T, \lambda, R)l.o.t.^z + \int_Q \lambda \{\mathcal{G}(z) + \mathcal{F}(z)\} \phi + C(E_z(T) + E_z(0)) \quad (3.11)$$

Proof. Taking into account (3.9) in (3.4), we have

$$\begin{aligned} \int_0^T \{ \|\Delta \phi\|^2 - \|\phi_t\|^2 \} &\leq C(T, \lambda)l.o.t.^z + \int_Q \lambda \{\mathcal{G}(z) + \mathcal{F}(z)\} \phi + \int_{\Sigma} B T^{\phi} \\ &\quad + \int_{\Sigma} \left\{ \partial_{\nu}(\Delta(\lambda z))(\lambda z) - \Delta(\lambda z) \partial_{\nu}(\lambda z) - \partial_{\nu}(\Delta z) \lambda^2 z \right. \\ &\quad \left. + 2\lambda z(\Delta z) \partial_{\nu} z + \lambda^2(\Delta z) \partial_{\nu} z \right\} + C(E(T) + E(0)) \end{aligned} \quad (3.12)$$

Taking into consideration boundary conditions (C) or (H) in (3.4), noting that $B.T^{\phi} = 0$ and accounting for the fact that the boundary terms resulting from the commutators vanish leads to the first statement in the Lemma. Calculations in the free case are analogous, and result from (3.5) and $\mathcal{B}_1 \phi = 0$, $\mathcal{B}_2 \phi = 2\beta \phi u w$, where the latter term contributes a lower order term to the estimate:

$$\left| \int_{\Gamma_1} \mathcal{B}_2 \phi \phi \right| \leq 2\beta \int_{\Gamma_1} |\phi|^2 |u| |w| \leq 2\beta R^2 \|\phi\|_1^2 \leq C(R)l.o.t.^z$$

□

3.1.2 Multiplier 2: $h \cdot \nabla \psi$

For the first part of this section, we specify only that $\text{supp } \mu \cap \Gamma = \emptyset$; otherwise, we keep μ as general as possible, specifying it at the last possible moment. Additionally, define a set $M \equiv \text{supp } \nabla \mu = \{x \in \Omega \mid \mu \not\equiv \text{constant}\}$. Now, if we multiply (3.2) by μ , and recall that $\psi \equiv \mu z$, we obtain

$$\psi_{tt} + \Delta^2 \psi + \mu \mathcal{G}(z) + \mu \mathcal{F}(z) = [\Delta^2, \mu] z$$

where $\mathcal{G}(z) = d(\mathbf{x}) (g(u_t) - g(w_t))$ and $\mathcal{F}(z) = -(f_V(u) - f_V(w))$, as before. We now make use of the multiplier $h \cdot \nabla \psi$, which we write as $h \nabla \psi$ henceforth; there are various choices for the vector field h , situationally dependent, however here we need only take $h = \mathbf{x} - \mathbf{x}_0 \in \mathbb{R}^2$ in order to obtain control on the potential energy of the plate. Now, as in the previous section, we multiply the last equality by our multiplier and use Green's Theorem to obtain

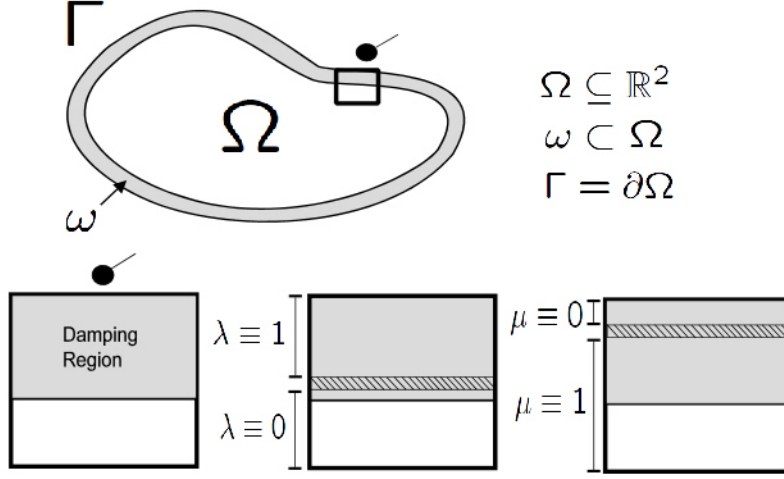
$$\int_Q (|\psi_t|^2 + |\Delta \psi|^2) \leq C(E_z(T) + E_z(0)) + \int_Q \mu \mathcal{G}(z) (h \nabla \psi) + \int_Q \mu \mathcal{F}(z) (h \nabla \psi) + \int_Q [\Delta^2, \mu] z (h \nabla \psi)$$

By explicitly writing out the commutator, and taking into account the support of $\nabla \mu$, upon splitting we obtain:

$$\int_Q [\Delta^2, \mu] z (h \nabla \psi) = \int_0^T \int_M [\Delta^2, \mu] z (h \nabla \psi) \leq C(\mu) \int_0^T \int_M |\Delta z|^2 + C(T, \mu)l.o.t.^z \quad (3.13)$$

Now, at this point we specify the specific structure of the supports for λ and μ (which up to now have

been general). The following picture illustrates our choice for these supports and their relationship to the damping region ω :



We emphasize that (a) the set $M \subset \text{supp } \lambda$ and (b) $\text{supp } \lambda$ and $\text{supp } \mu$ overlap inside the damping region ω and that $\text{supp } \lambda \cup \text{supp } \mu = \Omega$. Since $M \subset \{x \in \Omega : \lambda(x) \equiv 1\}$, we have the following inequality:

$$\begin{aligned}
 \int_Q [\Delta^2, \mu] z (h \nabla \psi) &\leq C(\mu) \int_0^T \int_M |\Delta z|^2 + l.o.t.^z \\
 &\leq C(\mu) \int_0^T \int_{\lambda \equiv 1} |\Delta z|^2 + C(\mu, T) l.o.t.^z \\
 &\leq C(\mu) \int_Q |\Delta \phi|^2 + C(\mu, T) l.o.t.^z
 \end{aligned} \tag{3.14}$$

3.2 Energy Recovery Estimate

We may now appeal to our calculation with the ϕ multiplier previously, to obtain our preliminary ψ estimate:

Lemma 3.2 (Preliminary ψ Estimate). *Let $\psi \equiv \mu z$, as defined above, where z solves (3.2) with any boundary conditions under considerations. Moreover, assume $\text{supp } (\mu)$ is bounded away from Γ . Then,*

in the case of clamped (C) or hinged (H) boundary conditions we have

$$\begin{aligned} \int_Q (|\psi_t|^2 + |\Delta\psi|^2) &\leq C(\mu, \lambda) \left\{ (E_z(T) + E_z(0)) + \int_Q \mu \{ \mathcal{G}(z) + \mathcal{F}(z) \} (h\nabla\psi) \right. \\ &\quad \left. + \int_Q \lambda \{ \mathcal{G}(z) + \mathcal{F}(z) \} \phi + C(T)l.o.t.^z \right\} \end{aligned} \quad (3.15)$$

In the case of free boundary conditions (F)

$$\begin{aligned} \int_0^T (||\psi_t||^2 + a(\psi, \psi)) &\leq C(\mu, \lambda) \left\{ (E_z(T) + E_z(0)) + \int_Q \mu \{ \mathcal{G}(z) + \mathcal{F}(z) \} (h\nabla\psi) \right. \\ &\quad \left. + \int_Q \lambda \{ \mathcal{G}(z) + \mathcal{F}(z) \} \phi + C(T)l.o.t.^z \right\} \end{aligned} \quad (3.16)$$

We note that in the nonlinear boundary term associated with the operator \mathcal{B}_2 vanishes due to the fact that the support of μ is away from the boundary.

We may now combine the estimates from Lemma 3.1 and Lemma 3.2 to obtain an estimate on the total energy (with either form of boundary conditions (C) or (H) or (F)):

$$\begin{aligned} \int_0^T \{ ||\psi_t||^2 + ||\phi_t||^2 + a(\phi, \phi) + a(\psi, \psi) + \beta \int_{\Gamma_1} |\phi|^4 \} &\leq C(\mu, \lambda) \left\{ (E_z(T) + E_z(0)) + \int_Q \mu \{ \mathcal{G}(z) + \mathcal{F}(z) \} (h\nabla\psi) \right. \\ &\quad \left. + \int_Q \lambda \{ \mathcal{G}(z) + \mathcal{F}(z) \} \phi + \int_0^T ||\phi_t||^2 + C(T, R)l.o.t.^z \right\} \end{aligned} \quad (3.17)$$

By our choice of supports for μ and λ we note that the LHS of the above equation *overestimates* the total energy $E_z(t)$. On the RHS of the estimate we have the term $\int_Q |\phi_t|^2$, which we replace by $\int_0^T \int_\omega |z_t|^2$ since $\text{supp } \lambda \subset \omega$ and on $\text{supp } \lambda$, $\lambda \leq 1$, so we have that

$$\int_Q |\phi_t|^2 \leq \int_0^T \int_\omega |z_t|^2$$

Making the appropriate changes above in Lemma 3.1 and Lemma 3.2, we have the analogous result for the free boundary conditions (F). Hence we can conclude

Lemma 3.3 (Preliminary Energy Estimate). *For any boundary condition (C), (H), or (F) we have*

$$\begin{aligned} \int_0^T E_z(t) &\leq C(\mu, \lambda) \left\{ (E_z(T) + E_z(0)) + \int_Q \mu \{ \mathcal{G}(z) + \mathcal{F}(z) \} (h\nabla\psi) + \int_Q \lambda \{ \mathcal{G}(z) + \mathcal{F}(z) \} \phi \right. \\ &\quad \left. + \int_0^T \int_\omega |z_t|^2 + C(T)l.o.t.^z \right\} \end{aligned} \quad (3.18)$$

Remark 3.2. At this point, we impose clamped (C) or hinged (H) boundary conditions, in order to simplify and streamline the analysis. At the end of this section, we discuss the boundary conditions (F).

If we take into account the supports of λ and μ (dropping dependence of the constants on μ, λ , and Ω) then (3.18) with clamped boundary conditions becomes

$$\begin{aligned} \int_0^T E_z(t) \leq & C \left\{ E_z(T) + E_z(0) + \int_Q \{ \mathcal{G}(z) + \mathcal{F}(z) \} (h \nabla \psi) + \int_Q \{ \mathcal{G}(z) + \mathcal{F}(z) \} z \right\} \\ & + C \int_0^T \int_\omega |z_t|^2 + C(T) l.o.t.^z \end{aligned} \quad (3.19)$$

Remark 3.3. At this point we pause to point out that the estimate we have shown above in (3.19) will be used in the sections to follow, specifically in the quasistability estimate. In particular, we must handle the damping terms (involving u_t, w_t) differently in the estimation for asymptotic smoothness, versus the estimation for quasistability.

By the assumptions on g in Assumption 1, for every δ there exists $C_\delta > 0$ such that

$$|u_t - w_t|^2 \leq \delta + C_\delta (g(u_t) - g(w_t)) (u_t - w_t).$$

This gives that

$$\int_0^T \int_\omega |z_t|^2 \leq T\delta|\Omega| + C(\delta) \int_0^T \int_\omega (g(u_t) - g(w_t)) z_t$$

or, simplifying, and taking into account $\omega \subset \text{supp } d$ and that $d(\mathbf{x}) \geq \alpha_0 > 0$, we have

$$\int_0^T \int_\omega |z_t|^2 \leq \delta + C(\delta, T, \Omega) \int_Q \mathcal{G}(z) z_t$$

So taking into account the last inequality in (3.19), we obtain

$$\begin{aligned} \int_0^T E_z(t) \leq & \delta + C \left\{ E_z(T) + E_z(0) + \int_0^T \int_\omega (\mathcal{G}(z) + \mathcal{F}(z)) z \right. \\ & + \int_Q (\mathcal{G}(z) + \mathcal{F}(z)) h \nabla \psi + C(\delta, T) \int_Q \mathcal{G}(z) z_t \left. \right\} \\ & + C(T) l.o.t.^z \end{aligned}$$

where the constant C does not depend on T . Recall, u and w are solutions to (1.1) corresponding to some initial conditions y_1 and y_2 , satisfying $S_t y_1 = (u(t), u_t(t))$ and $S_t y_2 = (w(t), w_t(t))$ for the evolution S_t associated to the plate dynamics. We can assume that $y_1, y_2 \in \mathcal{W}_R$ for some $R > R_*$, where the invariant set $\mathcal{W}_R = \{(u, v) \in \mathcal{H}, \mathcal{E}(u, v) \leq R\}$. Assuming the solutions u and w are strong, by the invariance of \mathcal{W}_R we have

$$\|u(t)\|_2 + \|u_t(t)\| + \|w(t)\|_2 + \|w_t(t)\| \leq C(R), \quad t \geq 0 \quad (3.20)$$

$$\|u(t)\|_{C(\Omega)} + \|w(t)\|_{C(\Omega)} \leq C(R), \quad t \geq 0 \quad (3.21)$$

Recent developments in the area of Hardy-Lizorkin spaces and compensated compactness methods allow one to show the following ‘sharp’ regularity of the Airy stress function v :

Theorem 3.4 (Sharp Regularity of the Airy Stress Function). [12]

$$\|v(u)\|_{W^{2,\infty}} \leq C\|u\|_2^2, \quad \|v(u, w)\|_{W^{2,\infty}} \leq C\|u\|_2\|w\|_2$$

where we have denoted $v(u, w) \equiv -\Delta^{-2}[u, w]$ and $\mathcal{D}(\Delta^2) = H^4(\Omega) \cap H_0^2(\Omega)$

Making use of the above inequalities, we have the estimate

$$\|[v(u), z]\| \leq C\|u(t)\|_2^2\|z\|_2 \leq C(R)\|z\|.$$

Additionally, we have

$$\|v(u) - v(w)\|_{W^{2,\infty}} = \|v(z, u + w)\|_{W^{2,\infty}} \leq C\|z\|_2(\|u\|_2 + \|w\|_2).$$

Therefore,

$$\|\mathcal{F}(z)\| = \|[u, v(u)] - [w, v(w)] + [z, F_0]\| = \|[v(u) - v(w), z] + [v(w), z] + [z, F_0]\| \leq C(R)\|z\|_2, \quad t \geq 0.$$

So we obtain

$$\int_0^T \int_\omega \mathcal{F}(z)z \leq \int_Q \mathcal{F}(z)z \leq \epsilon \int_0^T \|z(t)\|_2^2 dt + C(T, \epsilon)l.o.t.^z \quad (3.22)$$

and similarly

$$\int_Q \mathcal{F}(z)h\nabla\psi \leq C(R) \int_0^T \|z(t)\|_2\|\psi(t)\|_1 \leq \epsilon \int_0^T \|z(t)\|_2^2 + C(T, \epsilon)l.o.t.^z, \quad (3.23)$$

(where again, dependence of constants on Ω, ω , and h are suppressed). To proceed, we need estimates on the dissipation. By the energy equality

$$E_z(T) + \int_s^T \int_\Omega \mathcal{G}(z)z_t = E_z(s) + \int_s^T \int_\Omega \mathcal{F}(z)z_t, \quad (3.24)$$

we have

$$\int_Q \mathcal{G}(z)z_t \leq C(R) + \left| \int_Q \mathcal{F}(z)z_t \right| \quad (3.25)$$

Taking into account the embedding $H^{2-\eta}(\Omega) \subset C(\Omega)$ for $0 < \eta < 1$, we see

$$\begin{aligned} \int_Q \mathcal{G}(z)z &\leq \int_Q d(\mathbf{x})(|g(u_t)| + |g(w_t)|)|z| \\ &\leq C\|z\|_{C(0,T;C(\Omega))} \int_Q d(\mathbf{x})(|g(u_t)| + |g(w_t)|) \\ &\leq C\|z\|_{C(0,T;H^{2-\eta}(\Omega))} \int_Q d(\mathbf{x})(|g(u_t)| + |g(w_t)|). \end{aligned}$$

Splitting the region of integration according to $|u_t| \leq 1$ and $|u_t| > 1$, and similarly according to $|w_t| \leq 1$

and $|w_t| > 1$, we obtain

$$\int_Q d(\mathbf{x})(|g(u_t)| + |g(w_t)|) \leq g(1)\|d\|_{L^\infty(\Omega)}\text{meas}(Q) + \int_Q d(\mathbf{x})(g(u_t)u_t + g(w_t)w_t) \leq C(R, T)$$

Hence

$$\int_Q \mathcal{G}(z)z \leq C(R, T)l.o.t._1^z \quad (3.26)$$

Now applying Holder's inequality with the exponent $r > 1$ we see

$$\int_Q \mathcal{G}(z)h\nabla\psi \leq C \sup_{[0, T]} \|\nabla\psi(t)\|_{r'} \int_Q d(\mathbf{x})^r (|g(u_t)|^r + |g(w_t)|^r)$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Taking $r = 1 + \frac{1}{p+1}$, and again splitting the region of integration according to $|u_t| \leq 1$ and $|u_t| > 1$, and using the polynomial growth condition imposed on g in Assumption 1, we obtain

$$\int_Q d(\mathbf{x})^r |g(u_t)|^r \leq C(d)\{g(1)\text{meas}(Q) + \int_Q d(\mathbf{x})g(u_t)u_t\} \leq C(R)(T+1)$$

Since the same computations hold for terms in w , and we have the continuous embedding $H^{1-\delta}(\Omega) \hookrightarrow L_{r'}(\Omega)$ for sufficiently small δ , we have

$$\int_Q \mathcal{G}(z)h\nabla\psi \leq C(R, T)l.o.t._1^z \quad (3.27)$$

Hence by the above estimates, we have

$$\int_0^T E_z(t) \leq C\{E_z(T) + E_z(0) + \delta + C(R, \delta) + C(\delta) \int_Q \mathcal{F}(z)z_t + C(R, T)(l.o.t._1^z + l.o.t._1^z)\}$$

and eventually by (3.24) we have

$$\int_0^T E_z(t) \leq C_* \left\{ E(T) + \delta + C(R, \delta) + C(\delta) \left| \int_0^T \int_\Omega \mathcal{F}(z)z_t \right| + C(R, T)(l.o.t._1^z + l.o.t._1^z) \right\} \quad (3.28)$$

where we write C_* to emphasize that this constant *does not* depend on T . If we integrate (3.24) over $(0, T)$ with respect to the variable s , and take into account (3.28), we may choose T sufficiently large ($T > 2C_*$) and ϵ sufficiently small (with respect to T) such that

Lemma 3.5 (Asymptotic Smoothness Estimate).

$$E_z(T) \leq \epsilon + \frac{C(R, \epsilon)}{T} \left(1 + \left| \int_Q \mathcal{F}(z)z_t \right| + \left| \int_0^T \int_s^T \int_\Omega \mathcal{F}(z)z_t \right| \right) + C(\epsilon, R, T)(l.o.t._1^z + l.o.t._1^z) \quad (3.29)$$

We are now in a position to prove Theorem 1.3 on the existence of a compact attracting set \mathbf{A} . For this we shall invoke the abstract Theorem 2.2.

Completion of the proof of Theorem 1.3

To apply Theorem 2.2 we need to construct a functional $\Phi_{\epsilon,R,T}$ such that

$$\liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \Phi_{\epsilon,R,T}(y_n, y_m) = 0$$

for every sequence $\{y_n\}$ from B (following from Theorem 2.2). The functional will contain “noncompact and not-small” terms in the inequality (3.29). More specifically, for any initial data $U_0 = (u_0, u_1)$, $W_0 = (w_0, w_1) \in B$ we define

$$\tilde{\Phi}_{\epsilon,R,T}(U_0, W_0) = \left| \int_0^T (\mathcal{F}(z), z_t) \right| + \left| \int_0^T \int_t^T (\mathcal{F}(z), z_t) \right|$$

where the trajectory $z = u - w$ has initial data $U_0 - W_0$. The key to compensated compactness is the following representation for the bracket:

$$(\mathcal{F}(z), z_t) = \frac{1}{4} \frac{d}{dt} \{ -\|\Delta v(u)\|^2 - \|\Delta v(w)\|^2 + 2([z, z], F_0) \} - ([v(w), w], u_t) - ([v(u), u], w_t) \quad (3.30)$$

Integrating the above expression in time and evaluating on the difference of two solutions $z^{n,m} = w^n - w^m$, where $w^i \rightharpoonup w$ yields:

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T (\mathcal{F}(z^{n,m}), z_t^{n,m}) &= \frac{1}{2} \{ \|\Delta v(w)(t)\|^2 - \|\Delta v(w)(T)\|^2 \} \\ &\quad - \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T \{ ([v(w^n), w^n], w_t^m) + ([v(w^m), w^m], w_t^n) \}, \end{aligned} \quad (3.31)$$

where we have used (a) the weak convergence in $H^2(\Omega)$ of $z^{n,m}$ to 0, and (b) compactness of $\Delta v(w)$ from $H^2(\Omega) \rightarrow L_2(\Omega)$. The iterated limit in (3.31) is handled via iterated weak convergence, as follows:

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T \{ ([v(w^n), w^n], w_t^m) + ([v(w^m), w^m], w_t^n) \} \\ = 2 \int_0^T ([v(w), w], w_t) = \frac{1}{2} \|\Delta v(w)(t)\|^2 - \frac{1}{2} \|\Delta v(w)(T)\|^2. \end{aligned}$$

This yields the desired conclusion, that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T (\mathcal{F}(z^{n,m}), z_t^{n,m}) = 0.$$

The second integral term in $\tilde{\Phi}$ is handled similarly. As a consequence we obtain

$$\liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \tilde{\Phi}_{\epsilon,R,T}(y_n, y_m) = 0.$$

Now, we define

$$\Phi_{\epsilon,R,T} = \tilde{\Phi} + (l.o.t.^z + l.o.t.^z_1),$$

and noting that the terms $(l.o.t.^z + l.o.t.^z_1)$ in (3.29) are compact with respect to $H^2(\Omega)$ via the Sobolev

embeddings, the final conclusion follows by taking T sufficiently large and ϵ sufficiently small. This concludes the proof of smoothness estimate required by Theorem 2.2. Thus, the dynamical system is asymptotically smooth. In addition, stationary solutions are bounded (due to the inequality in Lemma 1.1) and the set $\{(u_0, u_1) \in \mathcal{H}, \mathcal{E}(u_0, u_1) \leq R\}$ is positively invariant. This implies ([12]) the existence of local attractors \mathbf{A}_R - the first statement in Theorem 1.3. For the second statement we just note that the UC property renders the dynamical system gradient with a strict Lyapunov function. Thus, asymptotic smoothness and boundedness of the set of stationary solutions implies the existence of a compact global attractor.

4 Regularity and finite dimensionality of the Attractor

Let \mathbf{A} be the global attractor corresponding to the flow S_t , as established in Section 3. To prove finiteness of the fractal dimension of the set \mathbf{A} , we shall use Theorem 2.3 which is based on a *quasistability* estimate. It will turn out that the quasistability estimate will follow from a more direct estimate written for the difference of two solutions $z = u - w$ at sufficiently *negative times*. Since the system is gradient, the latter due to the UC property assumed, we have that trajectories from the attractor stabilize asymptotically to equilibria points for both positive and negative times. This is to say: any $x \in \mathbf{A}$ belongs to some full trajectory $\gamma = \{(u(t), u_t(t)), t \in \mathbb{R}\}$ and for any $\gamma \subset \mathbf{A}$ there exists a pair $\{e, e^*\} \subset \mathcal{N}$ (set of equilibria) such that

$$(u(t), u_t(t)) \rightarrow (e, 0) \text{ as } t \rightarrow \infty, \quad (u(t), u_t(t)) \rightarrow (e^*, 0) \text{ as } t \rightarrow -\infty \quad (4.1)$$

4.1 Quasistability Estimate

We shall follow a general program developed in [14] and supported by PDE estimates derived in previous sections and specific to localized dissipation.

With the previous notation, we state the following lemma which gives a preliminary estimate for quasistability inequality:

Lemma 4.1. *Let $z \equiv u - w$ where $(u(t), u_t(t)), (w(t), w_t(t)) \in \mathbf{A}$ with $w(t) = u(t + h)$, $0 < h \leq 1$. Then, there exists $T_u \in \mathbb{R}$ such that for all $-\infty < s \leq s + T_0 \leq T_u$ the following inequality holds:*

$$E_z(s + T_0) + \int_s^{s+T_0} E_z(\tau) \leq C(\mathbf{A})T_0 D_s^{s+T_0} + C(\mathbf{A})T_0 \sup_{\tau \in [s, s+T_0]} \|z(\tau)\|_{2-\eta}^2 \quad (4.2)$$

for $\eta > 0$, where

$$D_{t_1}^{t_2} \equiv \int_{t_1}^{t_2} \int_{\Omega} d(\mathbf{x})(g(u_t) - g(w_t))z_t$$

This lemma will provide a tool for proving additional smoothness of the attractor. In fact, we will select a time T which will be negative (in line with the convergence in (4.1)). The above estimate, a posteriori, will yield the quasistability estimate in Theorem 2.3, and hence finite dimensionality of the attractor. More specifically, we shall follow the program described below after completing the proof of the above lemma: (1) Apply Lemma 4.1 with some negative time $s + T$, to conclude (by homogeneity) that the solution at T has additional derivatives - and is in fact a strong solution. For this step, we will

exploit closedness of velocities to a point of equilibria, along with a decomposition of the von Karman bracket, which allows us to take the advantage of this closedness. (2) The above gain of regularity is then propagated forward on the strength of regularity theory for von Karman evolutions. This gives an algebraic embedding for the attractor, but boundedness of the attractor in the higher norm will remain to be established; (3) This boundedness, that is, the topological embedding, will follow from a covering theorem, and the compactness of the attractor (shown in the earlier section). These details are given in [12]. Thus, at the end of this procedure we can extend validity of the inequality in Lemma 4.1 to all times $-\infty < s < t < \infty$, and we obtain the conclusion of the additional regularity of the attractor. This then allows us to infer the *quasistability* estimate in Theorem 2.3 (which is the same estimate as in Lemma 4.1, but valid for all times $s < t$ and all *pairs of trajectories* $(u(t), u_t(t))$ and $(w(t), w_t(t))$ lying in the attractor).

Thus the crux of the proof of regularity and dimensionality of the attractor reduces to the demonstration of Lemma 4.1. We also note that in comparison with the asymptotic smoothness inequality, the inequality in Lemma 4.1 is more demanding. This is due to necessity of keeping *at least quadratic forms* in the lower order terms. This very demand forces the damping to have at least linear growth at the origin $g'(0) > 0$. (Such restriction is typical - if not necessary - whenever regularity or finite dimensionality of attractors becomes a concern.)

4.2 Preparation for the Proof of Lemma 4.1

In the proof of Lemma 4.1, we will make use of the recovery estimate in (3.19) and the energy relation (3.24) as our main tools. Beginning with (3.19), and taking into account estimates involving $\mathcal{F}(z)$ in (3.22), (3.23) and the linear growth condition $g'(0) > 0$ in (3.19), we arrive at

$$\int_0^T \int_{\Omega} |z_t|^2 \leq CD_0^T(z)$$

Applying the above inequality in (3.19) gives:

$$\begin{aligned} \int_0^T E_z(\tau) \leq C \Big\{ D_0^T(z) + E_z(T) + E_z(0) + \left| \int_Q \mathcal{G}(z)z \right| \\ + \left| \int_Q \mathcal{G}(z)h\nabla z \right| \Big\} + C(R, T)l.o.t.^z \end{aligned} \quad (4.3)$$

Now, in tackling quadratic dependence of the dissipation terms, we give the following proposition

Proposition 4.2. *Let assumptions of Theorem 1.4 be satisfied, and take z be a solution to (3.2). Then there exists $\delta > 0$ such that*

$$\left| \int_Q \mathcal{G}(z)z \right| \leq \delta D_0^T(z) + C(\delta, R, T) \sup_{[0, T]} \|z\|_{2-\eta}^2, \quad 0 < \eta < 2 - \gamma \quad (4.4)$$

$$\left| \int_Q \mathcal{G}(z)h\nabla z \right| \leq \delta D_0^T(z) + C(\delta, R, T) \sup_{[0, T]} \|z\|_{2-\eta}^2, \quad 0 < \eta < 1 - \gamma \quad (4.5)$$

where $\mathcal{G}(z) = d(\mathbf{x})(g(u_t) - g(w_t))$ and $D_0^T(z) = \int_Q \mathcal{G}(z) z_t$.

Proof. We note that the assumptions on the damping function g (namely, monotonicity and the polynomial growth condition in Assumption 1) imply that

$$\frac{g(s_2) - g(s_1)}{s_2 - s_1} \leq C[1 + g(s_1)s_1 + g(s_2)s_2]^\gamma. \quad (4.6)$$

Using the Jensen inequality we estimate

$$|z| \leq \delta |z_t| + C(\delta) \frac{|z|^2}{|z_t|},$$

The above, along with (4.6), gives

$$\left| \int_Q \mathcal{G}(z) z \right| \leq \delta D_0^T(z) + C(\delta, M) \int_Q d(\mathbf{x}) (1 + (g_0(u_t)u_t)^\gamma + (g_0(w_t)w_t)^\gamma) |z|^2$$

Now, applying the Holder inequality with exponent $p = \gamma^{-1}$ and Sobolev's embedding $H^{2-\eta}(\Omega) \subset L_{\frac{2}{1-\gamma}}(\Omega)$, and taking into account energy equality (1.13) we arrive at

$$\left| \int_Q \mathcal{G}(z) z \right| \leq \delta D_0^T(z) + C(\delta, R) l.o.t.^z$$

The inequality in (4.5) can be shown analogously. \square

So, taking into account (4.4) and (4.5) in (4.3) we obtain

$$\int_0^T E_z(\tau) \leq C \left\{ D_0^T(z) + E_z(T) + E_z(0) \right\} + C(R, T) l.o.t.^z$$

We note that C does not depend on T , and $l.o.t.^z$ is of quadratic order. By using semigroup property and reiterating the same argument on the intervals $[s, s+T]$ one obtains

$$\int_s^{T+s} E_z(\tau) \leq C \left\{ D_s^{T+s}(z) + E_z(T+s) + E_z(s) \right\} + C(R, T) l.o.t.^z(s, T+s) \quad (4.7)$$

where $l.o.t.^z(s, T+s)$ denote lower order terms collected over the interval $[s, T+s]$.

In order to prove (4.2), we have to handle the non-compact term $(\mathcal{F}(z), z_t)$. A technical calculation based on the symmetry properties of von Karman bracket gives us the following proposition whose proof is given in [12].

Proposition 4.3. *If $u, w \in C([0, t]; H^2(\Omega)) \cap C^1([0, t]; L_2(\Omega))$ and $z = u - w$ then*

$$(\mathcal{F}(z), z_t) = \frac{1}{4} \frac{d}{dt} Q(z) + \frac{1}{2} P(z) \quad (4.8)$$

where

$$Q(z) = (v(u) + v(w), [z, z]) - \|\Delta v(u + w, z)\|^2$$

$$P(z) = -(u_t, [u, v(z, z)]) - (w_t, [w, v(z, z)]) - (u_t + w_t, [z, v(u + w, z)]). \quad (4.9)$$

Now, we can state the following lemma:

Lemma 4.4. *Let $u(\tau)$ and $w(\tau)$ be two functions from the class*

$$C([s, t]; H_0^2(\Omega)) \cap C^1([s, t]; L_2(\Omega))$$

for $s, t \in \mathbb{R}$, $s < t$, such that

$$\|u(\tau)\|_2^2 + \|u_t(\tau)\|^2 \leq R^2, \quad \|w(\tau)\|_2^2 + \|w_t(\tau)\|^2 \leq R^2, \quad \tau \in [s, t]$$

Let $z(\tau) = u(\tau) - w(\tau)$. Then for $\eta > 0$

$$\left| \int_s^t (\mathcal{F}(z), z_t) \right| \leq C(R) \sup_{\tau \in [s, t]} \|z(\tau)\|_{2-\eta}^2 + C(R) \int_s^t (\|u_t\| + \|w_t\|) \|z\|_2^2. \quad (4.10)$$

Proof. The inequality follows from the basic properties of von Karman bracket [12] and the decomposition in Proposition 4.3. \square

4.3 Completion of the Proof of Lemma 4.1.

Let $\gamma = \{(u(t), u_t(t)) : t \in \mathbb{R}\}$ be a trajectory from the attractor \mathbf{A} , and let $0 < h < 1$. It is clear that for the pair $w(t) \equiv u(t + h)$ and $u(t)$ satisfy the hypotheses of Lemma 4.4 for every interval $[s, t]$. We shall estimate the energy $E_z(t)$ of $z(t) \equiv z^h(t) = u(t + h) - u(t)$. Here we critically use the estimates for the noncompact term involving $\mathcal{F}(z)$. By (4.10), we have

$$\left| \int_s^t (\mathcal{F}(z^h), z_t^h) \right| \leq C(R) \sup_{\tau \in [s, t]} \|z^h(\tau)\|_{2-\eta}^2 + C(R) \int_s^t (\|u_t(\tau + h)\| + \|u_t(\tau)\|) \|z^h(\tau)\|_2^2. \quad (4.11)$$

for all $-\infty < s \leq t < +\infty$. Since we have the characterization $\mathbf{A} = M^u(\mathcal{N})$, where \mathcal{N} is the set of equilibria, we have

$$\lim_{t \rightarrow -\infty} d_{H_0^2(\Omega) \times L_2(\Omega)}(S_t W | \mathcal{N}) = 0 \quad \text{for any } W \in H_0^2(\Omega) \times L_2(\Omega)$$

Hence, for any $\epsilon > 0$, there exists T_γ^ϵ (independent of h but depending on the trajectory γ) such that

$$\|u_t(\tau)\| + \|u_t(\tau + h)\| \leq \epsilon [C(R)]^{-1} \quad \text{for any } t \leq T_\gamma^\epsilon.$$

Taking into account the last inequality in (4.11), we arrive at

$$\left| \int_s^t (\mathcal{F}(z^h), z_t^h) \right| \leq C(R) \sup_{\tau \in [s, t]} \|z^h(\tau)\|_{2-\eta}^2 + \epsilon \int_s^t \|z^h(\tau)\|_2^2. \quad (4.12)$$

for all $-\infty < s \leq t < T_\gamma^\epsilon$. Using the energy relation (3.24), we find from (4.12) that

$$E_z(s) \leq E_z(t) + \int_s^t \int_\Omega \mathcal{G}(z) z_t + C(R) \sup_{\tau \in [s, t]} \|z^h(\tau)\|_{2-\eta}^2 + \epsilon \int_s^t \|z^h(\tau)\|_2^2 d\tau. \quad (4.13)$$

and similarly

$$E_z(t) \leq E_z(s) + C(R) \sup_{\tau \in [s, t]} \|z^h(\tau)\|_{2-\eta}^2 + \epsilon \int_s^t \|z^h(\tau)\|_2^2. \quad (4.14)$$

for all $-\infty < s \leq t < T_\gamma^\epsilon$. Now, if we apply (4.7) on each subinterval $[s, s+T_0]$, we have

$$\int_s^{s+T_0} E_z(\tau) \leq C \left\{ D_s^{s+T_0}(z) + (E_z(s+T_0) + E_z(s)) \right\} + C(R, T_0) \sup_{\tau \in [s, s+T_0]} \|z(\tau)\|_{2-\eta}^2$$

Taking into account (4.13) in the last inequality and choosing ϵ sufficiently small we arrive at

$$\int_s^{s+T_0} E_z(\tau) \leq C \left\{ D_s^{s+T_0}(z) + E_z(s+T_0) + C(R, T_0) \sup_{\tau \in [s, s+T_0]} \|z(\tau)\|_{2-\eta}^2 \right\}$$

Now, integrating (4.14) and considering the previous inequality we have

$$E_z(s+T_0) + \int_s^{s+T_0} E_z(\tau) \leq C(\mathbf{A}, T_0) D_s^{s+T_0}(z) + C(\mathbf{A}, T_0) \sup_{\tau \in [s, s+T_0]} \|z(\tau)\|_{2-\eta}^2$$

which gives (4.2) and thus proves Lemma 4.1.

5 Proof of Theorem 1.4

5.1 Proof of Part (i) in Theorem 1.4

Having established Lemma 4.1 we now proceed with the proof of improved regularity for the attractor. This is done, as in [14], in three steps:

Step 1: Smoothness for negative times

By (4.2) and energy relation (3.24) written on the interval $[s, s+T_0]$ we can choose a constant $0 < \mu < 1$ such that $u^h(t) = h^{-1}z^h(t)$ satisfies the following estimate

$$E_{u^h}(s+T_0) \leq \mu E_{u^h}(s) + C(T_0) \sup_{\tau \in [0, T_0]} \|u^h(s+\tau)\|_{2-\eta}^2 \quad (5.1)$$

for all $s \leq T_\gamma - T_0$, where $T_\gamma = T_\gamma^{\epsilon_0}$ (depending on the trajectory, but not h) for some $\epsilon_0 > 0$ and $T_0 > 0$. Now using interpolation, and taking the supremum over the interval $(-\infty, T_\gamma - T_0)$ we obtain

$$\sup_{\tau \in (-\infty, T_\gamma]} E_{u^h}(\tau) \leq \frac{1+\mu}{2} \sup_{\tau \in (-\infty, T_\gamma]} E_{u^h}(\tau) + C(T_0)$$

This implies that

$$E_{u^h}(s) \leq C(T_0) \quad \text{for all } s \in (-\infty, T_\gamma] \quad (5.2)$$

After passing to the limit $h \rightarrow 0$ we get

$$\|u_{tt}(t)\|^2 + \|u_t(t)\|_2^2 \leq C \quad \text{for all } s \in (-\infty, T_\gamma] \quad (5.3)$$

By (5.3), for $u_t \in H^2(\Omega) \subset C(\Omega)$, we have $g(u_t) \in C(\Omega) \subset L_2(\Omega)$ by the continuity of g . Hence, elliptic regularity theory for $\Delta^2 u = -u_{tt} - d(x)g(u_t)$ with the boundary conditions gives that $\|u(t)\|_4^2 \leq C$ for all $t \in [-\infty, T_\gamma]$.

Step 2: Forward propagation of the regularity

Using the forward well-posedness of strong solutions stated in Theorem 1.2 (and the discussion that follows), we observe that $u(t)$ is a strong solution to the original problem, and so the global attractor \mathbf{A} is a subset of $(H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega)$.

Step 3: Boundedness of the attractor in $(H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega)$

In the previous step we have shown that $\mathbf{A} \subset (H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega)$. However, this does not guarantee the boundedness of \mathbf{A} in $(H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega)$. So, using the compactness of the attractor, we will follow an additional argument.

Since for every $\tau \in \mathbb{R}$, the element $u_t(\tau)$ belongs to a compact set in $L_2(\Omega)$ that consists of elements from $H_0^2(\Omega)$, for every $\epsilon > 0$ there exists a finite set $\{\phi_j\} \subset H_0^2(\Omega)$ such that we can find indices j_1 and j_2 such that

$$\|u_t(\tau) - \phi_{j_1}\| + \|u_t(\tau + h) - \phi_{j_2}\| \leq \epsilon.$$

Let $P(z)$ be given by (4.9) with the pair $w(t) = u(t + h)$ and $u(t)$ and

$$P_{j_1, j_2}(z) = -(\phi_{j_1}, [u, v(z, z)]) - (\phi_{j_2}, [w, v(z, z)]) - (\phi_{j_1} + \phi_{j_2}, [z, v(u + w, z)])$$

where $z(t) = u(t + h) - u(t) \equiv z^h(t)$. It can be easily shown that

$$\|P(z) - P_{j_1, j_2}(z)\| \leq \epsilon C(R) \|z^h(\tau)\|_2^2 \quad (5.4)$$

and

$$\|P_{j_1, j_2}(z)\| \leq C(R) (\|\phi_{j_1}\|_2 + \|\phi_{j_2}\|_2) \|z^h(\tau)\|_{2-\eta}^2$$

for $\eta > 0$. So we take

$$\sup_{j_1, j_2} \|P_{j_1, j_2}(z)\| \leq C(\epsilon) \|z^h(\tau)\|_{2-\eta}^2 \quad (5.5)$$

for $\eta > 0$. Taking into account (5.4) and (5.5) in (4.8) we see

$$\sup_{t \in [0, T]} \left| \int_{s+t}^{s+T} (\mathcal{F}(z^h), z_t^h) \right| \leq C(\epsilon, T, R) \sup_{\tau \in [0, T]} \|z^h(\tau + s)\|_{2-\eta}^2 + \epsilon \int_s^t \|z^h(\tau)\|_2^2$$

for all $s \in \mathbb{R}$ with $\eta > 0$ and $T > 0$. Now applying the above argument used to prove (5.1) and (5.2), we obtain the boundedness of the attractor \mathbf{A} in $(H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega)$.

This proves the first part of Theorem 1.4 .

5.2 Proof of (ii) in Theorem 1.4 - finite dimensionality

Using the compactness of the attractor and (5.4)-(5.5), we can write

$$\left| \int_s^{T+s} (\mathcal{F}(z), z_t) \right| \leq C(\epsilon)(1+T) \sup_{\tau \in [s, s+T]} \|z^h(\tau)\|_{2-\eta}^2 + \epsilon \int_s^{s+T} E_z(\tau)$$

for $s \in R$ and $\eta > 0$, where $z(t) = u(t) - w(t)$ with (u, u_t) and (w, w_t) from the attractor. Again, applying the same procedure from above, we obtain (5.1) for $u^h = z$ and $s \in \mathbb{R}$. We then note that (5.1) yields

$$E_z((m+1)T) \leq \gamma E_z(mT) + C(\mathbf{A}, T)b_m, \quad m = 0, 1, 2, \dots$$

with $0 < \gamma = \gamma(\mathbf{A}) < 1$, where

$$b_m \equiv \sup_{\tau \in [mT, (m+1)T]} \|z(\tau)\|^2$$

This yields

$$E_z(mT) \leq \gamma E_z(0) + c \sum_{l=1}^m \gamma^{m-l} b_{l-1}$$

Since $\gamma < 1$, there exist constants C_1, C_2 and σ possibly depending on R such that for all $t \geq 0$ we have

$$E_z(t) \leq C_1 E_z(0) e^{-\sigma t} + C_2 \sup_{\tau \in [0, t]} \|z(\tau)\|_{2-\eta}^2$$

which yields (2.2). Finally, on the strength of Theorem 2.3, applied with $B = \mathbf{A}$, $\mathcal{H} = D(\mathcal{A}^{1/2}) \times L_2(\Omega)$, $\mathcal{H}_1 = H^{2-\eta}(\Omega) \times \{0\}$ we conclude that \mathbf{A} has a finite fractal dimension.

References

- [1] P. Albana, Carleman estimates for the Euler-Bernoulli plate operator, *J. Diff. Eqns.*, v. 2000, 53, 2000, pp.1-13.
- [2] A. Babin, Global Attractors in PDE, In: B. Hasselblatt and A. Katok (ed.) Handbook of Dynamical Systems, v. 1B, *Elsevier Sciences*, Amsterdam, 2006.
- [3] A. Babin and M. Vishik, Attractors of Evolution Equations, *North-Holland*, Amsterdam, 1992.
- [4] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Nordhoff, 1976.
- [5] F. Bucci and D. Toundykov, finite dimensional attractor for a composite system of wave/plate equations with localized damping, *Nonlinearity*, 23, pp. 2271-2306, 2010.
- [6] M.S. Berger, On von Karman's equations and the buckling of a thin elastic plate, *Comm. Pure Appl. Math.*, 20, pp. 687-719, 1967.
- [7] I. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, *Acta*, Kharkov, 1999, in Russian; English translation: *Acta*, Kharkov, 2002.

- [8] I. Chueshov, Strong solutions and the attractors for von Karman equations, *Math USSR-Sb*, 69, 1991, pp. 25-36.
- [9] I. Chueshov, M. Eller, and I. Lasiecka, On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation, *Comm. in Partial Diff. Equations*, v. 27, pp. 1945-1948, 2002.
- [10] I. Chueshov and I. Lasiecka, long-time behavior of second-order evolutions with nonlinear damping, *Memoires of AMS*, v. 195, 2008.
- [11] I. Chueshov and I. Lasiecka, Attractors for second-order evolution equations with a nonlinear damping, *J. of Dyn. and Diff. Equations*, vol. 16, 2, 2004, pp.469-512.
- [12] I. Chueshov and I. Lasiecka, Von Karman Evolution Equations, *Springer-Verlag*, pp. 22, 44, 259, 2010.
- [13] I. Chueshov and I. Lasiecka, Global attractors for von Karman evolutions with a nonlinear boundary dissipation, *J. Diff. Eq.*, 198, 2004, pp. 196-231.
- [14] I. Chueshov and I. Lasiecka, Long-time dynamics of von Karman semi-flows with nonlinear boundary-interior damping. *J. Diff. Eq.*, 233, 2007, pp. 42-86.
- [15] I. Chueshov, I. Lasiecka, D. Toundykov, Global attractor for a wave equation with nonlinear localized boundary damping and a source term of critical exponent. *J. Dyn. Diff. Eq*, 21, pp. 269-314, 2009.
- [16] P. Ciarlet and P. Rabier, Les Equations de Von Karman, Springer, 1980.
- [17] A. Eden, C. Foias, B. Nicolaenko, and R. Temam, Exponential Attractors for Dissipative Evolution Equations, *Masson*, Paris, 1994.
- [18] M. Eller, V. Isakov, G. Nakamura, and D. Tataru, Uniqueness and stability in the Cauchy problem for Maxwell and elasticity system, *Stud. Math. Appl.*, v. 31, pp. 329-349, 2002.
- [19] M. Eller, Uniqueness of continuation theorems, In: R.P. Gilbert, et al. (eds) Direct and inverse problems of mathematical physics, 1st ISAAC Congress, Newark, DE, 1997, ISAAC5, *Kluwer*, Dordrecht, 2000.
- [20] V. Isakov, Inverse Problems for PDE's, *Springer Verlag*, 2006.
- [21] A. Favini, I. Lasiecka, M. A. Horn, and D. Tataru, Global existence, uniqueness and regularity of solutions to a von Karman system with nonlinear boundary dissipation, *Differential Equations 9 and 10*, 1996 and 1997, pp. 267-294 , pp. 197-200.
- [22] E. Fereisel, Attractors for wave equation with critical exponents and nonlinear dissipation. *C.R. Acad. Sci. Paris Ser I* 315, 1992, pp. 551-555
- [23] J. M. Ghidaglia and R. Temam, Regularity of the solutions of second order evolution equations and their attractors. *Ann Scuola Normal Superiore Pisa*, 14, 1987, pp. 485-511.

- [24] M. A. Horn and I. Lasiecka, Asymptotic behavior of Kirchoff plates, *J. Diff. Equations*, vol 11, pp.396-433, 1994.
- [25] M.A. Horn and I. Lasiecka, Uniform decay of weak solutions to a von Karman plate with nonlinear boundary dissipation, *Differential and Integral equations*, 7, 1994, 885-908.
- [26] V. Kalantarov, S. Zelik, Finite-dimensional attractors for the quasi-linear strongly-damped wave equation, *J. Diff. Eqs.*, v. 247, 4, 2009, pp. 1120-1155.
- [27] A.K. Khanmamedov, Global attractors for von Karman equations with non-linear dissipation, *J. Math. Anal. Appl*, 318, 2006, pp. 92-101.
- [28] H. Koch and I. Lasiecka, Hadamard wellposedness of weak solutions in nonlinear dynamic elasticity - full von Karman systems. *Evolution Equations, Semigroup and Functional Analysis*, vol 50, Birkhauser, 2002, pp. 197-212.
- [29] I. Lasiecka, Stabilization of wave and plate like equations with nonlinear dissipation on the boundary, *J. Diff. Equations*, vol. 79, pp. 340-381, 1989.
- [30] I. Lasiecka, Mathematical Control Theory of Coupled PDE's, CMBS-NSF Lecture Notes, *SIAM*, 2002.
- [31] I. Lasiecka, J.L. Lions, and R. Triggiani, Nonhomogeneous boundary value problems for second order hyperbolic operators, *J. Math. Pure et Appliques*, vol. 65, pp. 149-192, 1986.
- [32] I. Lasiecka and R. Triggiani, Trace regularity of the solutions of the wave equation with homogeneous Neumann boundary conditions and data supported away from the boundary, *J. of Math. Analysis and Appl.*, 141, 1989, pp. 49-71.
- [33] O. Ladyzhenskaya, Attractors for Semigroups and Evolution Equations, *Cambridge University Press*, Cambridge, 1991.
- [34] J. Lagnese, Boundary Stabilization of Thin Plates, *SIAM*, 1989.
- [35] J. Malek and D. Prazak, Large time behavior via the method of l-trajectories, *J. Diff. Eqs.*, v. 181, 2002, pp. 243-279.
- [36] S. Miyatake, Mixed problems for hyperbolic equations, *J. Math. Kyoto Univ.*, 13, 1973, pp. 435-487.
- [37] P. Cherrier and A. Milani, Parabolic equations of von Karman type on Kähler manifolds, II, *Bull. Sci. Math.*, v. 133, 2, 2009, pp. 113-133.
- [38] A. Miranville, S. Zelik, Attractors for dissipative partial differential equations in bounded and unbounded domains. In: M.C. Dafermos and M. Pokorný, (eds) Handbook of Differential Equations: Evolutionary Equations, v.4, *Elsevier*, Amsterdam, 2008.
- [39] A. Pazy, Semigroups of Linear Operators and Applications to PDE, *Springer*, New York, p 76, 1986.

- [40] D. Prazak, On finite fractal dimension of the global attractor for the wave equation with nonlinear damping, *J. Dyn. Diff. Eqs.*, v. 14, 2002, pp. 764-776.
- [41] G. Raugel, Global attractors in partial differential equations, In: Fiedler, B. (ed.) Handbook of Dynamical Systems, v. 2, *Elsevier Sciences*, Amsterdam, 2002.
- [42] R. Sakamoto, Mixed problems for hyperbolic equations, *J. Math Kyoto Univ.*, v. 2, 1970, pp. 349-373.
- [43] R.E. Showalter, Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations, v. 49, *AMS*, 1997.
- [44] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, *Springer-Verlag*, 1988.